Constructive Methods in Mathematics

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In Brief

The point of using constructive methods in mathematics is to explicitly exhibit any object or algorithm that the mathematician claims exists; so constructive proof provides, in principle, a mechanical method. Loosely speaking, one replaces the absolute notion of *truth* in mathematics, with (algorithmic) *provability*. Constructive proofs:

1. embody (in principle) an algorithm (for computing objects, converting other algorithms, etc.), and
2. prove that the algorithm they embody is correct (i.e. that it meets its design specification).

Constructive techniques

Upon adopting only constructive methods, we lose some powerful proof tools in our arsenal, such as unrestricted use of the Law of Excluded Middle (LEM) and anything which validates it, such as double negation elimination and unrestricted use of proof by contradiction[1]. We cannot, in general, constructively prove $\exists x P(x)$ by assuming $\neg\exists x P(x)$ and deriving a contradiction; that doesn't compute the required x.

However the news isn't all bad. In a lot of cases, constructive alternatives to non-constructive classical principles in mathematics, leading to some very strong results. For example, the classical least upper bound principle is not constructively provable.

LUB Any nonempty set of reals that is bounded from above has a least upper bound.

However the constructive least upper bound principle *is* provable.

CLUB Any order-located nonempty set of reals that is bounded from above has a least upper bound.

A set is order-located if given any real x, the distance from x to the set is computable. It is quite common for a constructive alternative to be *classically* equivalent to the classical principle; and, indeed, classically every nonempty set of reals is order-located.

To see why LUB is not provable, we may consider a so-called *Brouwerian counterexample* (or *weak* counterexample), such as the set

$$S = \{x \in \mathbb{R} : (x = 2) \vee (x = 3 \wedge P)\}$$

where P is some as-yet unproven statement, such as Goldbach's conjecture. If the set S had a computable LUB, then we would have a quick proof of the Goldbach conjecture's truth or of its unprovability. A Brouwerian counterexample is an example which shows that if a certain property holds, then it is possible

[1] Which is not to say that LEM is *false*. Both Russian recursive mathematics, in which LEM is provably false, and classical mathematics, in which it is logically true, are models of constructive mathematics—so in a way, LEM is *independent* of constructive mathematics, and hence non-constructive.

to constructively prove a non-constructive principle (such as LEM); and thus the property itself must be essentially non-constructive.

It is often the case that a classical theorem becomes more enlightening when seen from the constructive viewpoint[2]. For example, in the least upper bound principle the extra computational information provided by being order-located is enough to guarantee the computability of the least upper bound.

Within constructive mathematics a number of methods has been developed, enriching the subject to a degree where it is comparable to its classical counterpart in complexity, and often exceeds it in computational informativity.

Connections with other disciplines

The connection of constructive mathematics with computer science and programming is clear. A major upshot of the constructive approach is to identify with relative ease the sorts of things that computers cannot do (it is usually easier to prove a negative result), and so to guide the programmer to focus on what *is* achievable.

Like paraconsistency, constructivism brings out finer-grained details of proof that are often casually dismissed in classical proofs. In fact, a single classical theorem can lead to several constructively discernible *different* theorems, where the constructive techniques bring to the fore extra computational strength required in the hypotheses, or further information contained in the conclusion.

References

For a more in-depth introduction:

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Further reading:

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[2] Although it would be unfair to say that constructive mathematics is revisionist in nature. Indeed, Brouwer proved his fan theorem intuitionistically in 1927, but the first proof of König's lemma (its classical equivalent) was published in 1933.